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Weakly two-dimensional interaction of solitons in shallow water ☆

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Abstract

Nonlinear interactions of long-crested solitonic waves travelling in different directions in shallow water may serve as a possible mechanism of freak waves in certain sea areas. Several features of such interactions of equal amplitude Korteweg–de Vries (KdV) solitons in the framework of the Kadomtsev–Petviashvili equation are reviewed. In certain cases, nonlinear coupling produces a particularly high and steep wave hump. Interactions of equal amplitude solitons may lead to water surface elevations up to four times as high as the amplitude of the counterparts and the slope of the wave front may encounter eightfold increase. Exact expressions for the maximum slope in the case of interactions of unequal amplitude solitons are derived. The slope amplification for a certain class of interactions in the limiting case is twice as intense as the amplitude amplification. In the limiting case of exact resonance the interaction pattern is a new KdV soliton as in the case of the Mach stem. This feature allows to directly establish the extreme properties of the humps excited by nonlinear coupling. Evidence of such interactions and their possible consequences in realistic conditions are discussed.

Keywords: Extreme waves; Shallow water waves; Soliton interaction; Kadomtsev–Petviashvili equation

1. Introduction

Particularly high and steep (freak or rogue) waves on the sea surface are observed much more frequently than it might be expected from surface wave statistics. This interesting and dangerous phenomenon can be explained by various theories (see, for example, [1] and references therein). Stressing the importance of basically linear effects (currents, bathymetry, interaction with internal waves, etc. that may contribute to forming of freak waves, see [2–6] and bibliography therein), we focus here on a nonlinear mechanism of freak wave generation.

The idea that an appropriate nonlinear mechanism could be responsible for extreme waves has been employed in many studies (see recent overview of the relevant ideas in [1]). We concentrate here on a specific source for considerable changes in the wave amplitudes, namely, nonlinear interaction of shallow-water solitons. The most well-known soliton in such conditions has the shape of the solitary wave solution of the Korteweg–de Vries (KdV) equation and is called the KdV soliton in what follows.

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The KdV equation contains only one spatial dimension. The properties of a KdV soliton constant in the direction transversal to its propagation direction and it is usually considered as a (quasi)one-dimensional (1D) structure propagating along a channel of finite width. If excited in shallow water on an infinite plane, then a KdV soliton may propagate in any direction, and may meet analogous structures propagating in either the same or other directions. Interaction of unidirectional KdV solitons on such a plane is equivalent to their interaction in a channel and does not create any drastic increase of surface elevation [7]. Large elevations may occur when KdV solitons propagating in different directions meet each other [8,9]. The resulting structures can be described by means of multi-soliton solutions to the Kadomtsev–Petviashvili (KP) equation [10]. Since the KP equation can only be applied when the propagation directions of all the counterparts differ insignificantly, this framework may be called weakly two-dimensional interaction of solitons.

A fascinating manifestation of such an interaction is that in certain conditions a particularly high wave hump occurs in the vicinity of the crossing point of the soliton crests. As in the case of resonant interactions of Boussinesq solitons [11], the maximum elevation in the two-soliton solutions may be up to four times as high as the incoming waves. A part of the resulting pattern thus may be associated with rogue of freak water waves. Since the interaction pattern of two solitons has unlimited lifetime under favourable conditions, such a coupling may serve as one of the few mechanisms able to create long-living high waves in shallow water.

The interaction of two solitons is a generalisation of the Mach stem (also called Mach reflection, see, e.g. [11–14]). This mechanism is known in fluid dynamics for a long time for solitary waves reflecting from a wall [11,12] and apparently has already been observed by Scott Russell. It has an important application in coastal engineering [15,16] where it may result in extensive overtopping of breakwaters. It can be frequently observed when surface waves propagate in a channel [17] or in a semi-bounded area ([16,18], to mention a few studies). Although no extreme elevations were detected in early laboratory experiments [18], later experience confirms that this phenomenon does result in surface elevations much higher than the wave amplitude doubled (e.g. [19], Fig. 6).

The phenomena occurring during interaction of (trains of) solitonic waves in shallow water have been studied by several authors [9,20–22]. It has been only recently proposed as an explanation of the freak wave phenomenon [23]. The reason is that it may become evident only (i) provided long-crested shallow water waves can be associated with solitons and (ii) provided the KP equation is a valid model for such waves. These conditions are not common for storm waves; however, they may be often satisfied when two systems of swell are approaching a relatively shallow area from different directions. For example, a large part of the North Sea, in particular, the neighbourhood of the Dogger Bank, has a mean depth of 20–40 m. In this area, quite typical swell waves with a length over 100 m and a height of a few metres are effectively shallow water waves and a KdV or a KP equation is an acceptable model. Refraction of long-wave swell into a waveguide formed by spatially inhomogeneous currents may also lead to situation where groups of long-crested long waves cross each other under a reasonable angle and where contribution of nonlinear interactions may result in generation of freak waves inside the waveguide (see discussion in [24]).

The spatial extension of a high hump in the framework of soliton interactions is frequently associated with the area where the interacting waves have a common crest [23]. For equal amplitude incoming solitons the area where elevation exceeds the sum of amplitudes of the counterparts may considerably extend over the limits of this area [25]. The limits of the amplitude, the spatial occupancy of the high elevation and the slope of the front of the interaction pattern have been analysed in some detail for interactions of solitons of equal amplitude [23,25,26] that have specific symmetry properties.

Nonlinear interactions of solitons of unequal amplitudes are important in many applications (see, e.g. [8,11,21] and references therein). A part of the analysis of the properties of high elevations is extended to the case of interacting solitons with unequal amplitudes in [27]. The location and the height of the global maximum of the two-soliton solution as well as its symmetry properties are established in [27] for the case when the maximum amplitude exceeds the sum of amplitudes of the interacting solitons. The relative increase of the amplitude (compared to the sum of amplitudes of the incoming solitons) is largest when the counterparts are equal. Elevations greatly exceeding the sum of amplitudes of the counterparts only occur when the amplitudes of the intersecting solitons are close to each other.

A question of central interest is whether nonlinear interaction additionally affects the slope of the surface of multi-soliton solutions, because in realistic situations on sea not the heights of waves but their large slopes create acute danger. A pronounced feature of freak waves is that they are particularly steep. This feature is remarkably represented in the framework of the KP equation. Plots of two-soliton solutions in [21,23,26], suggest that the near-resonant high hump is particularly narrow and its front is very steep. The slope of the high wave hump may be up to 8 times as large as the slope of the incoming waves [25]. This feature can be recognised also in experiments with the use of the Mach reflection of supercritical ship wakes [17].

We extend the analysis of extremely large slopes of the nonlinear interaction pattern to the case of coupling of solitons with unequal amplitudes. A convenient co-ordinate system in phase variables, which makes use of certain symmetry properties of the two-soliton solutions, is introduced in Section 2. This section also recalls some earlier results which are now interpreted or proved in a more wide context. Section 3 contains derivation of exact expressions for the maximum front slope of the two-soliton solution for the case when its amplitude exceeds the sum of amplitudes of the incoming solitons. The new finding here consists in the proof that the previously known asymptotic estimate is an exact result in the proper co-ordinate system. In Section 4 it is shown how these expressions can be simply obtained in the exact resonance case, and the consistent evidence that the resonant structure is a KdV soliton is provided. A comparison of the magnitudes of amplitude and slope amplifications is presented in Section 5. Section 6 presents estimates of the spatial extent of the high wave hump. Since the results of Sections 5 and 6 can be obtained exactly in the same way as analogous results for equal amplitude solitons, only discussion of the results is presented. Potential circumstances of occurrence of such interactions in realistic conditions and their possible consequences are discussed in the final section.

2. Counterparts and heights of the two-soliton solution of the KP equation

We consider only two-soliton solutions of the KP equation, because structures arising in interactions of three and more solitons (although much higher than humps excited by interaction of two counterparts under certain conditions [28]) usually have short life-time. This section is based on our earlier results concerning the maximum height of the structure and is included because it is instructive to observe how the analytical expressions for the two-soliton solution can be greatly simplified in a proper co-ordinate system.

The standard Kadomtsev–Petviashvili equation in normalised variables (x, y, t, η) reads [29]

$$(\eta_t + 6\eta\eta_x + \eta_{xxx})_x + 3\eta_{yy} = 0, \tag{1}$$

where $\eta = \eta(x, y, t)$ has here the meaning of the elevation of the water surface, $\varepsilon = |\tilde{\eta}_{\max}|/H \ll 1$, and nondimensional variables are related to physical variables $(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\eta})$ as follows:

$$x = \sqrt{\varepsilon}(\tilde{x} - \tilde{t}\sqrt{gH})/H, \quad y = \varepsilon\tilde{y}/H, \quad t = \sqrt{\varepsilon^3 gH}\,\tilde{t}/H, \quad \eta = 3\tilde{\eta}/(2\varepsilon H) + \mathrm{O}(\varepsilon).$$

A two-soliton solution of Eq. (1) is $\eta = 2\partial^2 \ln(1 + e^{\varphi_1} + e^{\varphi_2} + A_{12}\, e^{\varphi_1+\varphi_2})/\partial x^2$, where $\varphi_i = k_i x + l_i y + \omega_i t$ are phase variables, $\kappa_i = (k_i, l_i)$, $i = 1, 2$, are the wave vectors of the incoming solitons (Fig. 1), the frequencies ω_i satisfy the dispersion relation $P(k_i, l_i, \omega_i) = k_i\omega_i + k_i^4 + 3l_i^2 = 0$ of the linearised KP equation, $A_{12} = -P(2k_-, 2l_-, \omega_1 - \omega_2)P^{-1}(2k_+, 2l_+, \omega_1 + \omega_2)$ is the phase shift parameter, $k_\pm = \frac{1}{2}(k_1 \pm k_2)$ and $l_\pm = \frac{1}{2}(l_1 \pm l_2)$ [21,22,29].

It is convenient to express the phase shift parameter as $A_{12} = [\lambda^2 - (k_1 - k_2)^2]/[\lambda^2 - (k_1 + k_2)^2]$, where $\lambda = l_1 k_1^{-1} - l_2 k_2^{-1}$ [21]. In the following we take $t = 0$ without loss of generality. Doing so is equivalent to introducing of

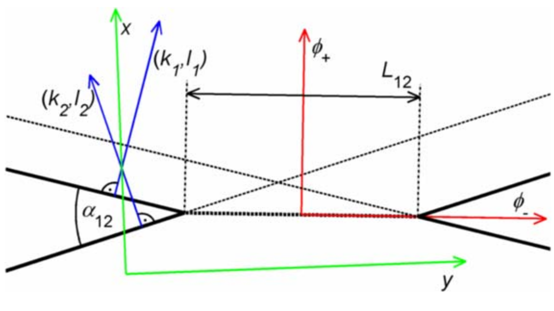

Fig. 1. Idealised patterns of crests of incoming solitons (bold lines, α_{12} is the angle between their propagation directions), their position in the absence of interaction (dashed lines) and the interaction soliton (bold dashed line) corresponding to the negative phase shift case in nondimensional physical (x, y) and phase variables (ϕ_+, ϕ_-) defined by Eq. (3). For better readability, the x- and y-axis are shown in a shifted location.

a properly moving co-ordinate system. This solution can be decomposed into a sum $\eta = s_1 + s_2 + s_{12}$ of two incoming solitons s_1, s_2 and residue (interaction soliton) s_{12} [22]:

$$s_{1,2} = A_{12}^{1/2} k_{1,2}^2 \Theta^{-2} \cosh\bigl(\varphi_{2,1} + \ln A_{12}^{1/2}\bigr), \quad s_{12} = \frac{1}{2} B \Theta^{-2}, \quad B = 4A_{12}k_+^2 + 4k_-^2,$$
$$\Theta = \cosh\bigl[(\varphi_1 - \varphi_2)/2\bigr] + A_{12}^{1/2} \cosh\bigl[(\varphi_1 + \varphi_2 + \ln A_{12})/2\bigr]. \tag{2}$$

The solution $\eta(x, y)$ is symmetric with respect to rotations by 180° around the point with co-ordinates $x_0 = l_-(k_1 l_2 - k_2 l_1)^{-1} \ln A_{12}$, $y_0 = -k_- l_-^{-1} x_0$ corresponding to $\varphi_1 = \varphi_2 = -\ln A_{12}^{1/2}$ and called the interaction centre. The maximum heights (amplitudes) $a_{1,2} = \frac{1}{2} k_{1,2}^2$ of the formal counterparts $s_{1,2}$ occur infinitely far from the interaction centre. This feature not necessarily reflects properties of realistic KdV solitons and their interactions, because the length of their crests is always limited (cf. Section 6), and the counterparts cannot be separated from each other and from the interaction soliton. If one of the counterparts has zero amplitude, the other one is the classical KdV soliton. The parameter $\Delta_{12} = -\ln A_{12}$ may be either positive or negative. In what follows we mostly consider the negative phase shift case $\Delta_{12} < 0$, $A_{12} > 1$ when $\max \eta \geqslant a_1 + a_2$.

In the case of equal amplitude solitons $k_1 = k_2$ with $l_1 = -l_2 = l$ both the whole pattern $\eta(x, y)$ and the interaction soliton s_{12} have two axes of symmetry, which are parallel to the co-ordinate axes and intersect at the interaction centre. The incoming solitons s_1, s_2 are the mirror images of each other with respect to these axes (Fig. 1). This symmetry is lost in interactions of solitons of unequal amplitudes. Yet the interaction soliton is always symmetric with respect to both the co-ordinate axes in the (ϕ_+, ϕ_-)-plane, where

$$\phi_+ = \frac{\varphi_1 + \varphi_2}{2} + \ln A_{12}^{1/2}, \qquad \phi_- = \frac{\varphi_1 - \varphi_2}{2}. \tag{3}$$

Other advantages of this co-ordinate system are that the interaction centre is at the origin and that expressions for s_1, s_2 and Θ can be greatly simplified [27]:

$$s_1 = A_{12}^{1/2} k_1^2 \Theta^{-2} \cosh(\phi_+ - \phi_-), \quad s_2 = A_{12}^{1/2} k_2^2 \Theta^{-2} \cosh(\phi_+ + \phi_-), \quad \Theta = \cosh \phi_- + A_{12}^{1/2} \cosh \phi_+. \tag{4}$$

The regular linear affine transformation defined by Eqs. (3) maps lines of the (x, y)-plane to lines of the (ϕ_+, ϕ_-)-plane (unless the wave vectors $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ are collinear). In particular, the lines $k_- x + l_- y = 0$ and $k_+ x + l_+ y + \ln A_{12} = 0$ correspond to the ϕ_+- and the ϕ_--axes, respectively. (This becomes clear from observation that for equal amplitude solitons the x-axis can be chosen so that it coincides with the ϕ_+-axes and the y-axis is parallel with the ϕ_--axis. The orientation of these axes in [27] may be misleading; however, the results described there are correct, because for stationary patterns and finite A_{12} the particular meaning of the axes is immaterial.) These lines are rectangular on the (x, y)-plane and serve as the pair of axes of symmetry of the interaction soliton only provided $|\boldsymbol{\kappa}_1| = |\boldsymbol{\kappa}_2|$ [27].

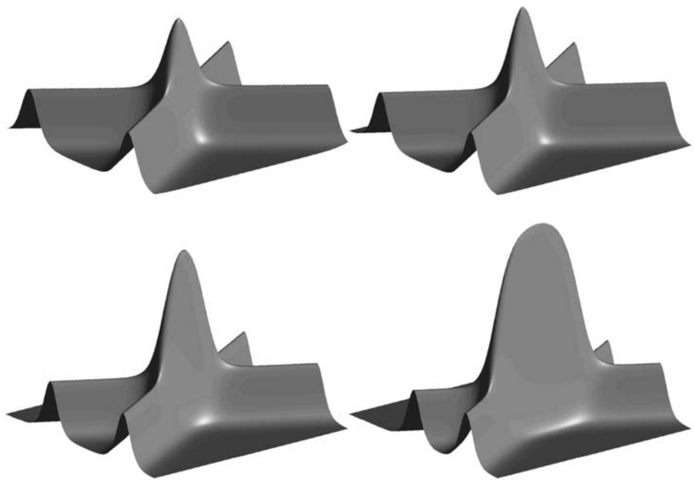

Fig. 2. Surface elevation in the vicinity of the interaction area, corresponding to incoming solitons with equal amplitudes $a_1 = a_2$, $l = -l_1 = 1/3$, $k_{\mathrm{res}} = \sqrt{1/3}$ and $k = 0.8k_{\mathrm{res}}$ (upper left panel), $k = 0.9k_{\mathrm{res}}$ (upper right), $k = 0.99k_{\mathrm{res}}$ (lower left), $k = 0.9999k_{\mathrm{res}}$ (lower right). Area $0 \leqslant z \leqslant 4a_1$, $|x| \leqslant 30$, $|y| \leqslant 30$ in normalised co-ordinates is shown at each panel (cf. [1,23]).

For the negative phase shift case ($A_{12} > 1$, typical in interactions of solitons with comparable amplitudes), the height of the interaction pattern exceeds the sum of the amplitudes of the two incoming solitons (Fig. 2, cf. also [11,12,30]). The maximum surface elevation for equal amplitude solitons is $a_{\max} = 4a_{1,2}/(1 + A_{12}^{-1/2})$ [11,25]. Thus, nonlinear superposition of two equal amplitude solitons may lead to a fourfold amplification of the surface elevation in the resonance case.

For unequal amplitude solitons the maximum elevation $a_{\max}$ for finite A_{12} and the amplitude of the resonant soliton a_∞ at $A_{12} = \infty$ are

$$a_{\max} = a_{12} + 2A_{12}^{1/2}\frac{a_1 + a_2}{(A_{12}^{1/2} + 1)^2}, \qquad a_\infty = \frac{(k_1 + k_2)^2}{2}. \tag{5}$$

The expression for a_∞ probably has been first obtained for exact resonance of ion-acoustic solitons in a field-free plasma [8] directly from the resonance conditions (assuming that the new structure is a KdV soliton) and re-derived from the conditions for stationary points of the explicit two-soliton solution of the KP equation in [31]. A simple derivation of expressions (5) without employing the assumption of the existence of the resonant soliton is given in [27].

3. Slope of the front of the two-soliton solution

Another important property the interaction pattern is the maximum slope of its front. It is of particular interest when the function $\eta = \eta(x, y)$ has the meaning of the elevation of the water surface in rough seas. The steepest slope of long-crested (resp. weakly 2D) waves is usually found roughly perpendicular to the direction of wave crests, equivalently, roughly parallel with the wave propagation direction. The slope in the latter direction is particularly important in applications of the extreme wave theory, because this is the slope of the wave which hits vessels or offshore structures.

A wave crest can be heuristically defined as a set of points corresponding to the maximum of the wave profile in the direction of its propagation. Since the counterparts of the two-soliton solution propagate at different directions, this definition is ambiguous. Moreover, the formal crests of the incoming solitons form quite a complex pattern in the process of soliton interaction [25].

Since the interaction patterns in question are weakly 2D structures, the crests of the counterparts and of the composite structure are roughly perpendicular to the ϕ_+-axis. Therefore, the steepest descent apparently exists roughly along the ϕ_+-direction. Based on this feature, a first approximation of the location of all the crests in question have been established in [23,25] from the analysis of maxima of elevations either in the ϕ_+-direction or in the principal propagation direction, which is roughly parallel to the ϕ_+-axis. More generally, (e.g. if the propagation direction is not known or there exists only a snapshot of the water surface), crests of a smooth surface $\eta(x, y)$ could be defined as lines of curvature [32] corresponding to the minimum normal curvature of the surface and going through a maximum (minimum) of this surface [27].

The maximum slope of the front of the interaction pattern of two equal amplitude solitons, asymptotically, may become as large as 8 times the maximum slope of the counterparts [25]. The correctness of the asymptotic procedure remained unclear, because the point, in which the maximum slope was to occur, was infinitely far from the origin of the co-ordinate system used in [25]. The co-ordinate system on the (ϕ_+, ϕ_-)-plane is more convenient for analysis of the extreme slopes and allows consistent derivation of estimates for the slopes occurring in interactions of unequal amplitude solitons as well.

Let us look for the maximum slope of the solution containing unequal amplitude solitons along the ϕ_+-axis. The procedure has much in common with the asymptotic analysis presented [25]. For brevity we omit the details of derivation and discussion of these equations, which can be restored from [25], and only refer to the relevant equations in [25].

The slopes of the counterparts in the ϕ_+-direction are

$$\begin{aligned} \frac{\partial s_{12}}{\partial \phi_+} &= -A_{12}^{1/2} B \Theta^{-3} \sinh\phi_+, \\ \frac{\partial s_{1,2}}{\partial \phi_+} &= A_{12}^{1/2} k_{1,2}^2 \Theta^{-3}\big[\Theta \sinh(\phi_+ \mp \phi_-) - 2\cosh(\phi_+ \mp \phi_-) A_{12}^{1/2} \sinh\phi_+\big], \end{aligned} \tag{6}$$

where the upper sign corresponds to s_1 and the lower sign to s_2. The maximum slope of one of the counterparts can be calculated directly from the expression for the soliton solution of the KdV equation; however, it is more consistent to start from expressions (2) for the two-soliton solution.

If only one soliton is present (say, $k_2 = s_2 = 0$) and propagates in the direction of the x-axis ($l_1 = l_2 = \lambda = 0$), then $A_{12} = 1$, $\Theta = 2\cosh\frac{1}{2}k_1x$, $s_2 = s_{12} = k_1^2\Theta^{-2}$ and $\eta = \frac{1}{2}k_1^2\cosh^{-2}\frac{1}{2}k_1x$. It thus can be identified as linear superposition of identical structures s_1 and s_{12}, which together form an infinitely long-crested solitary wave with the constant profile of a classical KdV soliton [7] along the crest. This wave obviously is a single KdV soliton. Transformation (3) is formally incorrect because the zero wave vector $\boldsymbol{\kappa}_2 = (k_2, l_2)$ can be interpreted as collinear with $\boldsymbol{\kappa}_1$. However, it is easy to see that $\phi_+ = \frac{1}{2}\phi_1 = \frac{1}{2}k_1x$, $\eta = \frac{1}{2}k_1^2\cosh^{-2}\phi_+$ and that the structure $s_1 + s_{12}$ is constant in the transversal direction. Thus the (ϕ_+, ϕ_-)-plane is just the (x, y)-plane stretched in the x-direction by the factor $\frac{1}{2}k_1$.

The slope $S^1 = -\frac{1}{2}k_1^2\sinh\phi_+\cosh^{-3}\phi_+$ of this single KdV soliton has a maximum value

$$S_{\max}^{1,2} = \frac{k_{1,2}^2}{3\sqrt{3}} \tag{7}$$

when $|\sinh\phi_+| = \sqrt{1/2}$, $\cosh\phi_+ = \sqrt{3/2}$. Note that expression (7) gives the slope of the wave front on the (ϕ_+, ϕ_-)-plane, which is generally stretched along both the axes as compared to the (x, y)-plane. For that reason the maximum slope, which is proportional to the amplitude cubed on the (x, y)-plane [25], is proportional to the amplitude squared in (7). Such a difference is immaterial in the analysis of the changes of the slope.

The slope of the interaction pattern is

$$\begin{aligned} S = \frac{A_{12}^{1/2}}{\Theta^3}\big\{&k_1^2\big[\Theta\sinh(\phi_+ - \phi_-) - 2A_{12}^{1/2}\cosh(\phi_+ - \phi_-)\sinh\phi_+\big] \\ &+ k_2^2\big[\Theta\sinh(\phi_+ + \phi_-) - 2A_{12}^{1/2}\cosh(\phi_+ + \phi_-)\sinh\phi_+\big] - B\sinh\phi_+\big\}. \end{aligned} \tag{8}$$

Since the global maximum of the interaction pattern for the negative phase shift case is at the interaction centre [27], the largest slope of the front apparently exists near this point. For that reason, we only consider the slope $\widetilde{S}$ along the ϕ_+-axis. Equivalently, we have $\phi_- = 0$ and expression (8) can be simplified as follows:

$$\widetilde{S} = \frac{A_{12}^{1/2}}{\Theta_0^3}\big[-B + \big(k_1^2 + k_2^2\big)\big(1 - A_{12}^{1/2}\cosh\phi_+\big)\big]\sinh\phi_+, \quad \Theta_0 = 1 + A_{12}^{1/2}\cosh\phi_+. \tag{9}$$

The slope $\widetilde{S}$ obviously is zero at $\phi_+ = 0$. For any other ϕ_+ where $\widetilde{S} = 0$ we have

$$A_{12}^{1/2}\cosh\phi_+ = 1 - \frac{B}{k_1^2 + k_2^2}. \tag{10}$$

For the negative phase shift case $A_{12} > 1$ Eq. (10) has no real solutions. Note that the situation is different in the positive phase shift case when for a certain range of A_{12} there exist two additional points of zero slope, symmetric to each other with respect to the ϕ_--axis. For equal amplitude solitons $k_1 = k_2$, Eq. (10) reads $A_{12}^{1/2}\cosh\phi_+ = 1 - 2A_{12}$ and has real solutions if $A_{12} < 1/4$ [25].

The location of the maximum slope in the ϕ_+-direction can be found from the condition $\partial S/\partial\phi_+ = 0$. From (6) we have general expressions for slopes of the counterparts and the interaction soliton:

$$\begin{aligned} \frac{\partial^2 s_{12}}{\partial\phi_+^2} &= \frac{A_{12}^{1/2}B}{\Theta^4}\big(3A_{12}^{1/2}\sinh^2\phi_+ - \Theta\cosh\phi_+\big), \\ \frac{\partial^2 s_{1,2}}{\partial\phi_+^2} &= \frac{A_{12}^{1/2}k_{1,2}^2}{\Theta_4}\big[-4\Theta A_{12}^{1/2}\sinh(\phi_+ \mp \phi_-)\sinh\varphi_+ - 2A_{12}^{1/2}\Theta\cosh(\phi_+ \mp \phi_-)\cosh\phi_+ \\ &\quad + 6A_{12}\cosh(\phi_+ \mp \phi_-)\sinh^2\phi_+ + \Theta^2\cosh(\phi_+ \mp \phi_-)\big]. \end{aligned} \tag{11}$$

At the φ_+-axis, $\Theta = \Theta_0 = 1 + A_{12}^{1/2}\cosh\varphi_+$ and the condition $\partial S/\partial\varphi_+ = 0$ can be written as

$$\begin{aligned} &A_{12}\big(k_1^2 + k_2^2\big)\cosh^3\varphi_+ + 2A_{12}^{1/2}\big[B - 2\big(k_1^2 + k_2^2\big)\big]\cosh^2\varphi_+ \\ &\quad + \big[(1 - 2A_{12})\big(k_1^2 + k_2^2\big) - B\big]\cosh\varphi_+ + A_{12}^{1/2}\big[4\big(k_1^2 + k_2^2\big) - 3B\big] = 0. \end{aligned} \tag{12}$$

This cubic equation with respect to $\cosh\phi_+$ is a generalisation of Eq. (29) in [25]. Important information about its real solutions (which require $\cosh\phi_+ \geqslant 1$) can be found based on the sum of all its coefficients. This sum $(1 - 3A_{12}^{1/2} - A_{12})(k_1^2 + k_2^2) - A_{12}^{1/2}B - B$ is negative for the negative phase shift case $A_{12} > 1$, and may become positive only for very small A_{12}. Therefore, there exists always at least one solution $\cosh\phi_+ \geqslant 1$ corresponding to the maximum slope provided $A_{12} > 1$. Physically, the existence of such a solution is obvious, because the slope of the surface is zero at the interaction centre, then increases in both directions of the x-axis, and approaches zero at infinity.

In the particular case of no phase shift $A_{12} = 1$, Eq. (10) can be reduced to $\cosh^3\phi_+ - 3\cosh\phi_+ - 2 = 0$ and has an obvious solution $\cosh_1\phi_+ = 2$, $\sinh_1\phi_+ = \sqrt{3}$ (cf. [25]). This case is equivalent to the linear superposition of the counterparts, because neither phase shift nor changes in the resulting wave amplitude occur; however, this is possible only if one of the incoming waves is infinitesimally small [21]. The maximum slope in this case can be found from a generalisation of expression (7):

$$\widetilde{S}_{\max} = \frac{1}{3\sqrt{3}}\left(k_1^2 + k_2^2\right). \tag{13}$$

Interaction of exactly unidirectional waves with $l_1 = l_2 = 0$ actually implies $A_{12} < 1$ [23] and the height of the resulting structure does not exceed the sum of the heights of the counterparts and extreme waves do not occur.

For near-resonant case $A_{12} \to \infty$, Eq. (12) has an asymptotic solution $\cosh_\infty\phi_+ = \sqrt{3/2}$, $\sinh_\infty\phi_+ = \sqrt{1/2}$, which coincides with the analogous solution for the single solitons. The slope at this point at the resonance case $A_{12} \to \infty$ is

$$\widetilde{S}_\infty = \frac{2(k_1 + k_2)^2}{3\sqrt{3}}. \tag{14}$$

Expression (14) is the generalisation of an analogous result for equal amplitude solitons [25]. In [25] the corresponding point at the y-axis is located at the distance $\sim \ln A_{12}^{1/2}$ from the origin. This location tends to infinity when $A_{12} \to \infty$ and the maximum slope calculated in [25] is, strictly speaking, correct only asymptotically. An important advantage of the co-ordinate system used in the current paper is that the point where the maximum slope occurs is located at a finite distance from the origin. A minor disadvantage is that the (ϕ_+, ϕ_-)-plane has a different scale compared to the (x, y)-plane. Therefore, slope described by expression (14) may be formed in realistic conditions in some vicinity of the interaction centre.

4. Resonant structures

Formally, the basic result of the previous section – expression (14) – can be found directly from the assumption that in the limiting case $A_{12} \to \infty$ the composite structure (or a part of it) is a KdV soliton (cf. [8]), frequently called resonant soliton. Since the resonant soliton concentrates energy of the incoming solitons into one structure, its amplitude and wavenumber are $\frac{1}{2}(k_1 + k_2)^2$ [8,27] and $k_\infty = k_1 + k_2$, respectively. Its maximum slope on the (x, y)-plane can be now found from Eq. (26) of [25]:

$$S_{\max}^{\infty} = \frac{k_\infty^3}{3\sqrt{3}} = \frac{(k_1 + k_2)^3}{3\sqrt{3}}. \tag{15}$$

For equal amplitude solitons, the slope of the resulting structure is $2^3 = 8$ times as steep as the slopes of the incoming solitons. Note that this relation cannot be directly obtained on the (ϕ_+, ϕ_-)-plane, on which the structure is stretched out in the ϕ_+-direction. Therefore, such an extreme slope amplification is not unexpected. It basically reflects the fact that the maximum slope of the KdV solitons increases as their heights cubed.

Heuristically, it is consistent that the resonant structure belongs to the same class of solutions as the incoming solitons (e.g. [8,11]). This hypothesis has been partially confirmed by proving that a specific asymptotic limit of the resulting structure at the exact resonance case satisfies the same equation as the incoming solitons. Although this is an important step towards establishing the nature of the resonant structure, strictly speaking, this result does not prove that the resonant soliton is a soliton of the same type as the interacting solitons.

The (near-)resonant cases have been usually considered as ‘three arms’ structures (e.g. [7, pp. 196–197; 11,12,33]), consisting of two incoming arms and a (near-)resonant structure, and resembling the Mach stem. The viewpoint is usually set in the vicinity of the bifurcation area of the 'arms' (e.g. [7,33]) and the whole interaction process occurs on a half-plane. The limiting process $A_{12} \to \infty$, result of which gives information about the properties of the resonant structure, is equivalent to moving the reflecting wall into infinity. After performing this operation, the analysis of phase variables in another limiting process (approaching to infinity towards this wall along the common crest of the Mach stem, usually performed with the use of phase variables) confirms that the resulting structure is a KdV soliton [33]. The 'truly' resonant structure, however, becomes evident infinitely far from the viewpoint, and only asymptotic estimates for its structure can be obtained (cf. [25]).

The described concept, which is equivalent to reducing of a limit of a function of several variables to a specific iterated limit, cannot be directly extended to realistic soliton interactions on a full plane. The obvious reason is that the length of the crests of the incoming solitons is limited and, consequently, the resulting structure is finite. This structure is also symmetric with respect to the interaction centre, which is equivalent to the crossing point of the crest of the Mach stem and the reflecting wall. As different from the Mach stem, there is always an elevation pattern at the other side of the interaction centre (resp. reflecting wall). This symmetry means that proper information about the resonant structure can only be obtained when the two above-mentioned limiting processes are synchronised so that the result reflects the properties of the resulting structure at the interaction centre. The results obtained in this way are useful but, strictly speaking, can be only accepted as indicative.

The structure of the interaction pattern at $A_{12} \to \infty$ can be analysed in a more consistent manner with the use of the (ϕ_+, ϕ_-)-co-ordinates employed in [27] and in the current study. Their basic advantage is that the origin (viewpoint) is at the interaction centre and the interaction pattern is treated as an 'four arms' X-like structure as in [9,20]. In the near-resonant case, the 'arms' representing the incoming solitons only occur at a large distance from the origin and the 'visible' structure in the vicinity of the origin represents the near-resonant wave hump. This feature allows to find the properties of the resulting structure as a simple limit of the relevant profile when $A_{12} \to \infty$.

The profile of the composite structure near the origin can be obtained from expressions (2), (4) by setting $\phi_- = 0$. This yields $\Theta = 1 + A_{12}^{1/2} \cosh\phi_+$, $s_{1,2} = A_{12}^{1/2} k_{1,2}^2 \Theta^{-2} \cosh\phi_+$, $s_{12} = 4(A_{12}k_+^2 + k_-^2)\Theta^{-2}$, and

$$\eta = s_1 + s_2 + s_{12} = \left[A_{12}^{1/2}\left(k_1^2 + k_2^2\right)\cosh\phi_+ + 4\left(A_{12}k_+^2 + k_-^2\right)\right]\Theta^{-2}. \tag{16}$$

The profile of this structure at the exact resonance

$$\eta_{\lim +} = \lim_{A_{12}\to\infty} \frac{4k_+^2 + A_{12}^{-1/2}(k_1^2 + k_2^2)\cosh\phi_+ + 4A_{12}^{-1}k_-^2}{\cosh^2\phi_+ + 2A_{12}^{-1/2}\cosh\phi_+ + A_{12}^{-1}} = (k_1 + k_2)^2 \cosh^{-2}\phi_+ \tag{17}$$

is that of the KdV soliton with the wave vector $k_\infty = k_1 + k_2$ and amplitude a_∞ (Eq. (5)). Its profile in the transversal direction (resp. along its crest) is, as expected, constant. This can be easily checked by setting $\phi_+ = \phi_+^* = const$ in expressions (2), (4) and taking the limit

$$\eta_{\lim -} = \lim_{A_{12}\to\infty,\phi_+=\phi_+^*} \eta = \frac{(k_1 + k_2)^2}{\cosh^2\phi_+^*} = const. \tag{18}$$

Therefore, the central part of the composite structure becomes the KdV soliton with the amplitude of a_∞ in the limit $A_{12} \to \infty$.

5. Amplitude and slope amplification factors

For interaction of unequal amplitude solitons, the maximum water elevation in the case of their (unrealistic) linear superposition obviously is the sum of the heights of the counterparts. When two waves of arbitrary amplitudes a_1 and a_2 meet, the maximum amplitude M of their superposition can be written as $M = m_{\text{amp}}(a_1 + a_2)$, where the 'amplitude amplification factor' m_{amp} may depend on both a_1 and a_2 and their intersection angle. This factor

$$m_{\text{amp}} = \frac{(k_1 + k_2)^2}{k_1^2 + k_2^2} \tag{19}$$

reaches 2 for interactions of equal amplitude solitons, is close to 2 when the amplitudes of the interacting solitons differ insignificantly and is close to 1 when they are fairly different [27].

The analysis above and in [25] shows that the maximum slope of the wave front increases even more drastically compared to the amplification of amplitudes owing to nonlinear interaction of solitons. The maximum slope of the water surface in the case of their (also unrealistic) linear superposition cannot be easily defined, because superposition of even unidirectional KdV solitons has quite a complex profile. If we assume that the maximum slopes of the counterparts are simply added during such a superposition (this is correct for equal amplitude solitons as well as for linear waves of equal length), the corresponding limit of the 'slope amplification factor' can be obtained from Eqs. (7) and (14):

$$m_s = \frac{2(k_1 + k_2)^2}{k_1^2 + k_2^2} = 2m_{\text{amp}}. \tag{20}$$

This factor in the limiting case is exactly twice the analogous amplitude amplification factor [27], reaches the maximum value $m_s = 4$ in the case of equal amplitude solitons, and is close to 1 for interactions of solitons with greatly different amplitudes.

6. Extent of high wave humps for unequal amplitude solitons

The spatial extent of the extreme slopes apparently follows the extent of the extreme elevations. The limits of the latter area are studied in [25] for the special case of equal amplitude solitons. To the first approximation, the area where the two-soliton solution exceeds the amplitude occurring in the process of linear superposition of s_1 and s_2 can be well described with the use of the (geometric) length L_{12} of the idealised common part (Fig. 1) of the crests of the incoming soliton [23,25]. This length is proportional to $\ln A_{12}$ and therefore is modest unless the interacting solitons are near-resonant. Although the area where surface elevation exceeds the sum of amplitudes of the counterparts may outstrip the estimates based on the geometry of the wave crests [25], the length of this area also is roughly proportional to $\ln A_{12}$.

The co-ordinates for description of the interaction pattern can be always chosen so that $l_1 = -l_2 = l$. The pattern is not necessarily steady in such co-ordinates, however, the geometric length $L_{12} = l^{-1} \ln A_{12}$ [23] only depends on the l-component of the wave vectors and on the phase shift parameter. Therefore, the spatial extent of appearance of nonlinear effects for interactions of solitons with drastically different amplitudes apparently is roughly as large as if the amplitudes were equal (Fig. 3). This feature may be important in applications where the extent and orientation of the near-resonant structure are equally important [1,8,9,21–23]. For example, near-resonant interaction of solitonic surface waves with radically different amplitudes in shallow sea areas may become evident in the form of bending of crests of the waves [9,21,31] rather than in the form of extreme elevations (Fig. 3).

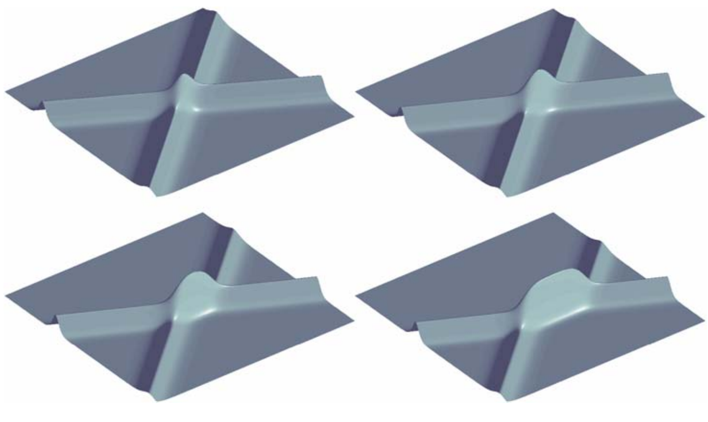

Fig. 3. Surface elevation in the vicinity of the interaction area, for $k_2 = 1/3$, $l = -l_1 = 0.2$, $k_{\text{res}} = 0.6$ and $k_1 = 0.9k_{\text{res}}$ (upper left panel), $k_1 = 0.99k_{\text{res}}$ (upper right), $k_1 = 0.999k_{\text{res}}$ (lower left), $k_1 = 0.9999k_{\text{res}}$ (lower right) in normalised co-ordinates (x, y). Area $|x| \leqslant 60$, $|y| \leqslant 90$ is shown at each panel.

7. Soliton interactions a source of freak waves in realistic conditions

The above has shown that the particularly high wave hump resulting from interaction of KdV solitons has a considerable length only when the heights of the incoming solitons, their intersection angle and the local water depth are specifically balanced. Consequently, the fraction of sea surface occupied by extreme elevations is apparently small as compared with the area of solitonic wave trains. However, a wave hump from such a nonlinear interaction, theoretically, has unlimited life-time and may cross large sea areas in favourable conditions [1]. Thus, one should account for the expected life-time of nonlinear wave humps when estimating the probability of occurrence of abnormally high waves.

A pronounced feature of freak waves is that they are particularly steep. According to some authors, the shape of the factually measured freak waves cannot be explained with the use of the existing wave physics whereas "it is concluded that new physics, not incorporated in standard approaches to offshore engineering design, may have played an important role in the generation of this [Draupner's New Year 1995] freak wave" [34]. The effect of nonlinear interactions in the framework of the KP equation that substantially modifies the profile of the two-soliton solution may be a part of this new physics for relatively shallow areas. The extraordinary steepness of the front of the near-resonant hump is a specific manifestation of nonlinear interaction between two solitary structures that shrinks the width of the joint structure. This result, although intriguing, is not unexpected, because the resonant KdV soliton is higher and therefore narrower than the incoming solitons.

The steepness of even quite high solitonic waves is moderate for the depth $h \approx 70$ m of the Draupner area. The dimensional profile of solitary waves is $\tilde{\eta} = a\cosh^{-2}(\beta x)$ where a is the amplitude of the soliton and $\beta \approx \frac{1}{2}\sqrt{3a/h^3}$ [7]. The maximum steepness of 3, 5 and 8 m high solitary waves for this depth is 0.006, 0.013 and 0.026, respectively. However, the maximum steepness of a 18 m high solitary wave, which may be formed through (un)favourable interaction of two long and long-crested waves with a height of about 5 m, would be about 0.087. This is somewhat less than the factual slope of the front of the Draupner wave.

The heuristically obvious result that the resonant structure is a perfect KdV soliton partially answers the question: what happens if the central part of the near-resonant interaction pattern hits a harbour entrance or a channel, or a river mouth [25]? It will probably move on as a single entity, which concentrates the energy of both the counterparts in one structure. Such wave hump may easily break before it reaches its theoretically maximum height, or after it propagates into an area where the conditions for existing of the two-soliton solution are not satisfied [23]. The possibility of breaking of the high and nonlinear wave hump makes a hit by a near-resonant structure exceptionally dangerous. For largely different amplitudes of the interacting solitons, the amplitude amplification remains modest and the interaction mostly leads to bending of the crests of both the counterparts. This effect may lead to hits by high waves arriving from an unexpected direction.

The process of nonlinear interaction of truncated (semi-infinite) waves with sech^2 profile, height of which varies along their crest, has been recently studied numerically in [35]. Interaction of waves, parameters of which correspond to an existing two-soliton solution, leads to structures similar to these solutions. For a certain range of parameters of incoming waves the interaction pattern is completely different from the two-soliton solution and to some extent resembles asymmetric Mach stem for Stokes waves [36]. The major features of the interaction in question are that (i) the wave amplitudes behind the interaction region are less than the amplitudes of the incoming solitons and that (ii) the wave crests before and after interaction are no more parallel.

A long-living wave hump, height of which considerably exceeds the sum of the heights of the counterparts, is formed quite fast in certain interaction region. As different from the Mach reflection of Stokes waves [36], it apparently tends to a stationary state [24] although such an exact state was not reached in the experiments. Its maximum height probably does not exceed the fourfold amplitude of the incoming solitons as in the case of the two-soliton solution. Since the initial waves are effectively two-dimensional, transversal energy flow along the crests supposedly occurs and the results are not directly comparable with the those obtained in the analysis of the two-soliton solution.This also applies to the results obtained for interaction of waves with curved crests [35] where local geometrical focusing drastically increases the amplitude of the wave hump.

Similar experiments have been performed, with results analogous to the described ones, in another weakly 2D framework – the modified KP (mKP) equation in which the quadratic term of the KP equation is replaced by cubic term

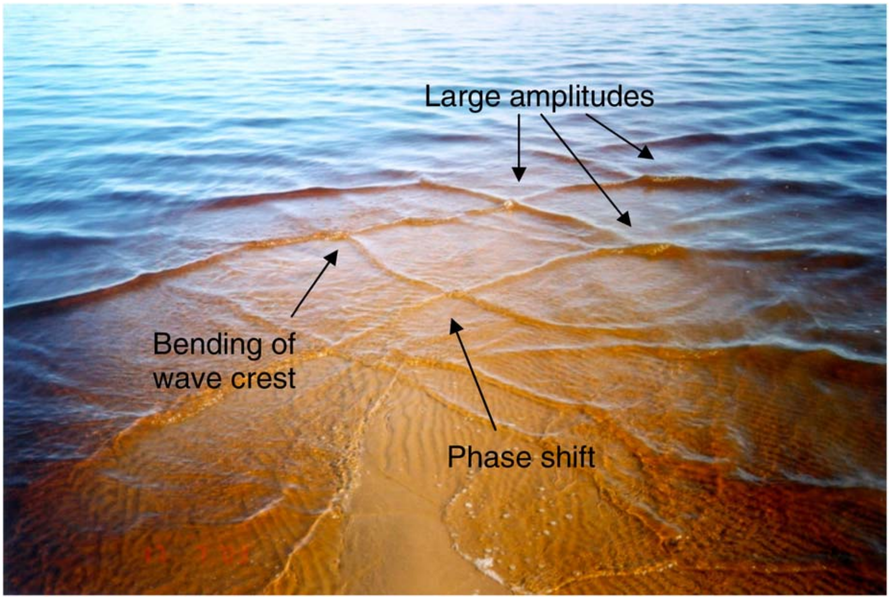

Fig. 4. Interaction patterns of solitonic surface waves in very shallow water near Kauksi resort on Lake Peipsi, Estonia. Photo by Lauri Ilison, July 2003.

$6\eta^2\eta_x$ [37]. This suggests that formation of intense and spatially concentrated structures in weakly 2D interactions of solitons is a more general phenomenon than just forming of freak waves on the water surface.

For completeness of discussion, we note that another class of exact solutions to the KP equation, allowing effects of soliton interactions and potential amplitude amplification, has been recently found [38]; however, the analysis of relevant extreme elevations has not been performed yet.

It is not clear whether the above-discussed features can be recognised in isolated form in open sea conditions. There is, however, increasing evidence that they may become evident under some specific conditions. For example, many of the analysed properties of nonlinear interaction are frequently observed in very shallow water (Fig. 4).

The performed analysis has an intriguing application in maritime security and coastal engineering in areas adjacent to intense high-speed ship traffic. The reason for that are trains of long-crested solitonic disturbances resembling Korteweg–de Vries (KdV) solitons that are frequently excited by contemporary ships if they sail at speeds roughly equal with the maximum phase speed of gravity waves [39–44]. Another source of solitonic waves is the classical Kelvin wake, long components of which may obtain the shape of KdV solitons in shallow areas (Soomere et al. [45]).

High solitonic ship wakes may cause considerable remote impact of the ship traffic owing to their low decay rates and their exceptional compactness [46]. To the knowledge of authors, they were first characterised as “an unknown phenomenon” in [47,48]. Their interaction with sea bottom and with each other may be responsible for dangerous waves along shorelines. It is no more unusual that holidaymakers are forced to ‘flee for their lives when enormous waves erupted from a millpond-smooth sea’, or that waves (that caused a fatal accident near Harwich in July 1999) look like ‘the white cliffs of Dover’ (Hamer [49]).

Groups of solitonic waves intersecting at a small angle may appear if wakes from two ships meet each other. Extreme surface elevations occur if the solitons intersect under a physical angle $\tilde{\alpha}_{12} = 2\arctan\sqrt{3\tilde{\eta}/h}$. This angle is reasonable (about 36°) for waves with heights $\tilde{\eta}_{\max} = 1.8$ m (the maximum documented ship wave height in Tallinn Bay [50]) meeting each other in an area with a depth of $h = 50$ m. However, such solitary waves are so gently sloping that even the resonant composite structure has the maximum slope of 0.036, which is hardly distinguishable on open sea. The situation may be different in shallow coastal areas. The critical angle (about 70°) apparently is out of the range of validity of the KP equation for frequently occurring solitonic ship waves with heights of $\tilde{\eta} = 0.8$ m and maximum slopes of 0.043 in the coastal zone with a depth of 5 m [50]. Crossing of such waves may, at least theoretically, produce extremely steep structures which have a height over 3 m, a maximum slope of about 0.34, and which are close to breaking.

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